احسب المجاميع التالية :

$$S_2=\frac{2}{11}+\frac{3}{11^2}-\frac{5}{11^3}+\frac{2}{11^4}+\frac{3}{11^5}-\frac{5}{11^6}+\dots\ ;\ S_1=\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+\frac{1}{7^5}+\frac{2}{7^6}+\dots$$

$$S_4=\lim_{n\to+\infty}\sum_{k=1}^{k=n}\frac{k}{4^k}\qquad ;\qquad S_3=\lim_{n\to+\infty}\sum_{k=1}^{k=n}\frac{k}{\pi^{k-1}}$$

.1

$$S_1=\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+\frac{1}{7^5}+\frac{2}{7^6}+\dots=\left(\frac{1}{7}+\frac{1}{7^3}+\frac{1}{7^5}+\dots\right)+\left(\frac{2}{7^2}+\frac{2}{7^4}+\frac{2}{7^6}+\dots\right)$$

$$=\lim_{n\to+\infty}\frac{1}{7}.\frac{1-\left(\frac{1}{7}\right)^{2n}}{1-\left(\frac{1}{7}\right)^2}+\lim_{n\to+\infty}\frac{2}{7^2}.\frac{1-\left(\frac{1}{7}\right)^{2n}}{1-\left(\frac{1}{7}\right)^2}=\frac{1}{7}.\frac{7^2}{7^2-1}+\frac{2}{7^2}.\frac{7^2}{7^2-1}=\dots=\frac{3}{16}$$

ومنه : $\boxed{S_1=\frac{3}{16}}$

طريقة ثانية مختصرة :

$$S_1=\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+\frac{1}{7^5}+\frac{2}{7^6}+\dots=\left(\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\frac{1}{7^4}+\frac{1}{7^5}+\frac{1}{7^6}+\dots\right)+\left(\frac{1}{7^2}+\frac{1}{7^4}+\frac{1}{7^6}+\dots\right)$$

$$=\lim_{n\to+\infty}\frac{1}{7}.\frac{1-\left(\frac{1}{7}\right)^{n}}{1-\left(\frac{1}{7}\right)}+\lim_{n\to+\infty}\frac{1}{7^2}.\frac{1-\left(\frac{1}{7}\right)^{2n}}{1-\left(\frac{1}{7}\right)^2}=\dots=\frac{1}{7}.\frac{7}{6}+\frac{1}{7^2}.\frac{7^2}{7^2-1}=\dots=\frac{3}{16}$$

.2

$$S_2=\frac{2}{11}+\frac{3}{11^2}-\frac{5}{11^3}+\frac{2}{11^4}+\frac{3}{11^5}-\frac{5}{11^6}+\dots=\left(\frac{2}{11}+\frac{2}{11^4}+\frac{2}{11^7}+\dots\right)+\left(\frac{3}{11^2}+\frac{3}{11^5}+\frac{3}{11^8}+\dots\right)$$

$$-\left(\frac{5}{11^3}+\frac{5}{11^6}+\frac{5}{11^9}+\dots\right)=\lim_{n\to+\infty}\frac{2}{11}.\frac{1-\left(\frac{1}{11}\right)^{3n}}{1-\left(\frac{1}{11}\right)^3}+\lim_{n\to+\infty}\frac{3}{11^2}.\frac{1-\left(\frac{1}{11}\right)^{3n}}{1-\left(\frac{1}{11}\right)^3}-\lim_{n\to+\infty}\frac{5}{11^3}.\frac{1-\left(\frac{1}{11}\right)^{3n}}{1-\left(\frac{1}{11}\right)^3}$$

أي : $S_2=\frac{2}{11}.\frac{11^3}{1330}+\frac{3}{11^2}.\frac{11^3}{1330}-\frac{5}{11^3}.\frac{11^3}{1330}=\left(\frac{2}{11}+\frac{3}{11^2}-\frac{5}{11^3}\right)\frac{11^3}{1330}=\dots=\frac{27}{133}$

ومنه : $\boxed{S_2=\dfrac{27}{133}}$

3. الكتابة $S_3=\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{k=n}\dfrac{k}{\pi^{k-1}}$ تعني : $S_3=\lim\limits_{n\to+\infty}\dfrac{1}{\pi^0}+\dfrac{2}{\pi^1}+\dfrac{3}{\pi^2}+\ldots+\dfrac{n}{\pi^{n-1}}$

أي : $S_3=\lim\limits_{n\to+\infty}1+2\left(\dfrac{1}{\pi}\right)^1+3\left(\dfrac{1}{\pi}\right)^2+\ldots+n\left(\dfrac{1}{\pi}\right)^{n-1}$

من الشكل : $S_3=\lim\limits_{n\to+\infty}1+2x+3x^2+\ldots+nx^{n-1}$ حيث : $x=\dfrac{1}{\pi}$

بوضع : $f(x)=1+2x+3x^2+\ldots+nx^{n-1}$

يكون : $\forall x\in\mathbb{R}-\{1\}:\int f(x)dx=x+x^2+x^3+\ldots+x^n+k=x\dfrac{x^n-1}{x-1}+k=\dfrac{x^{n+1}-x}{x-1}+k$

وبالتالي : $f(x)=\dfrac{d}{dx}\left(\dfrac{x^{n+1}-x}{x-1}+k\right)=\ldots=\dfrac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$

أي : $1+2x+3x^2+\ldots+nx^{n-1}=\dfrac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$ بتعويض $x=\dfrac{1}{\pi}$

ينتج : $1+2\left(\dfrac{1}{\pi}\right)+3\left(\dfrac{1}{\pi}\right)^2+\ldots+n\left(\dfrac{1}{\pi}\right)^{n-1}=\dfrac{n\left(\dfrac{1}{\pi}\right)^{n+1}-(n+1)\left(\dfrac{1}{\pi}\right)^n+1}{\left(\left(\dfrac{1}{\pi}\right)-1\right)^2}$

ومنه : $S_3=\lim\limits_{n\to+\infty}\dfrac{n\left(\dfrac{1}{\pi}\right)^{n+1}-(n+1)\left(\dfrac{1}{\pi}\right)^n+1}{\left(\left(\dfrac{1}{\pi}\right)-1\right)^2}=\dfrac{1}{\left(\dfrac{1}{\pi}-1\right)^2}=\dfrac{\pi^2}{(\pi-1)^2}$

ومنه : $\boxed{S_3=\dfrac{\pi^2}{(\pi-1)^2}}$

برهان : $\lim\limits_{n\to+\infty}(n+1)\left(\dfrac{1}{\pi}\right)^n=0\ ;\ \lim\limits_{n\to+\infty}n\left(\dfrac{1}{\pi}\right)^{n+1}=0$ يعتمد على : $\lim\limits_{n\to+\infty}\dfrac{n}{a^n}$

انظر البرهان

4. الكتابة $S_4=\lim\limits_{n\to+\infty}\sum_{k=1}^{k=n}\dfrac{k}{4^k}$ تعني : $S_4=\lim\limits_{n\to+\infty}\dfrac{1}{4}+\dfrac{2}{4^2}+\dfrac{3}{4^3}+\ldots+\dfrac{n}{4^n}$

أي : $S_4=\lim\limits_{n\to+\infty}1\left(\dfrac{1}{4}\right)+2\left(\dfrac{1}{4}\right)^2+3\left(\dfrac{1}{4}\right)^3+\ldots+n\left(\dfrac{1}{4}\right)^n$

من الشكل : $S_4=\lim\limits_{n\to+\infty}x+2x^2+3x^3+\ldots+nx^n$ حيث : $x=\dfrac{1}{4}$

<u>طريقة أولى :</u> بوضع : $g(x)=x+2x^2+3x^3+\ldots+nx^n$

يكون : $g(x)=x.f(x)=\dfrac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$ أنظر f المذكورة في الفرع 3

أي : $x+2x^2+3x^3+\ldots+nx^n=\dfrac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$

<u>طريقة ثانية</u>: بوضع : $g(x)=x+2x^2+3x^3+\ldots+nx^n\ \ldots(1)$

يكون : $x.g(x)=x^2+2x^3+3x^4+\ldots+nx^{n+1}\ldots(2)$ ،

بطرح (2) من (1) نجد : $(1-x)g(x)=x+x^2+x^3+\ldots+x^n-nx^{n+1}$

أي : $(1-x)g(x)=x.\dfrac{1-x^n}{1-x}-nx^{n+1}=\dfrac{x-x^{n+1}-nx^{n+1}+nx^{n+2}}{1-x}=\dfrac{x-(n+1)x^{n+1}+nx^{n+2}}{1-x}$

أي : $x+2x^2+3x^3+\ldots+nx^n=\dfrac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$

بتعويض $x=\dfrac{1}{4}$ ينتج : $\dfrac{1}{4}+\dfrac{2}{4^2}+\dfrac{3}{4^3}+\ldots+\dfrac{n}{4^n}=\dfrac{n\left(\dfrac{1}{4}\right)^{n+2}-(n+1)\left(\dfrac{1}{4}\right)^{n+1}+\dfrac{1}{4}}{(\dfrac{1}{4}-1)^2}$

ومنه : $S_4=\lim\limits_{n\to+\infty}\dfrac{n\left(\dfrac{1}{4}\right)^{n+2}-(n+1)\left(\dfrac{1}{4}\right)^{n+1}+\dfrac{1}{4}}{(\dfrac{1}{4}-1)^2}=\dfrac{\dfrac{1}{4}}{\dfrac{9}{4^2}}=\dfrac{4}{9}$

ومنه : $S_4=\dfrac{4}{9}$

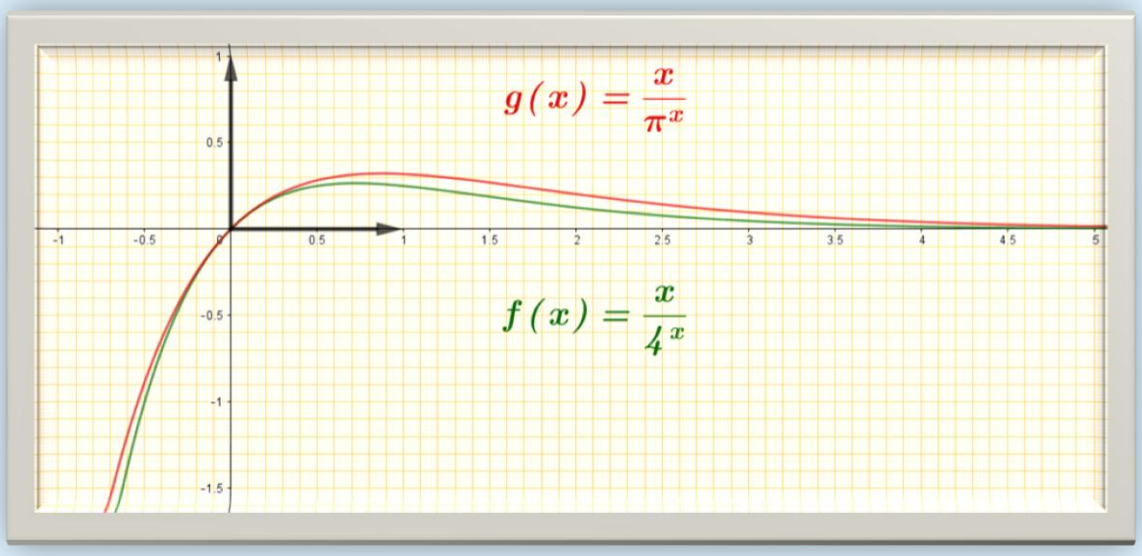
برهان : $\lim_{n\to+\infty}(n+1)\left(\frac{1}{4}\right)^{n+1}=0$; $\lim_{n\to+\infty} n\left(\frac{1}{4}\right)^{n+2}=0$ يعتمد على : $\lim_{n\to+\infty}\frac{n}{a^n}$

$$\lim_{n\to+\infty}\frac{n}{a^n}$$

$$\forall a\in\mathbb{R}_+^*-\{1\}:\lim_{n\to+\infty}\frac{n}{a^n}=\lim_{n\to+\infty}\frac{n}{e^{n\ln a}}=\lim_{n\to+\infty}\frac{1}{\ln a}.\frac{n\ln a}{e^{n\ln a}}$$

$$=\begin{cases}\frac{1}{\ln a}.\lim_{x\to-\infty}\frac{x}{e^x}=+\infty & si\ \ 0<a<1\\ \frac{1}{\ln a}.\lim_{x\to+\infty}\frac{x}{e^x}=0 & si\ \ a>1\end{cases}\qquad tq\ \ x=n\ln a$$

ومنه : $\lim_{n\to+\infty}\frac{n}{4^n}=0$; $\lim_{n\to+\infty}\frac{n}{\pi^n}=0$



لا تترددوا بملاحظاتكم واقتراحاتكم